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The structure of non-nilpotent CTI-groups

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Abstract. A subgroup H of a group G is called a TI-subgroup if $H \cap H^g \in \{1, H\}$, for all $g \in G$, and a group is called a CTI-group if all of its cyclic subgroups are TI-subgroups. In this paper, we determine the structure of non-nilpotent CTI-groups. Also we will show that if G is a nilpotent CTI-group, then G is either a Hamiltonian group or a non-abelian p-group.

1 Introduction and preliminaries

Throughout the following, G always denotes a finite group.

Let H be a subgroup of G. If for every $g \in G$ we have $H \cap H^g \in \{1, H\}$, then H is called a TI-subgroup. Now if every subgroup of G is a TI-subgroup, then G is called a TI-group, and G is an ATI-group if all of its abelian subgroups are TI-subgroups. In [13], G. Walls classified the TI-groups. S. Li and X. Guo in [6] classified the ATI-groups of prime power order; also these authors with P. Flavell in [4] determined the structure of ATI-groups.

A subgroup H of G is called a QTI-subgroup if for every $1 \neq x \in H$, we have

$$\mathcal{C}_G(x) \leq \mathcal{N}_G(H).$$

A group G is called a QTI-group if all of its subgroups are QTI-subgroups; correspondingly, G is an AQTI-group if all its abelian subgroups are QTI-subgroups. It can be shown that any TI-subgroup is a QTI-subgroup, but the converse is not true. In [8], G. Qian and F. Tang classify AQTI-groups and prove that if G is a p-group, then the properties of being TI, ATI and AQTI are equivalent in G.

Groups all of whose cyclic subgroups are TI-subgroups are called CTI-groups. Clearly, any ATI-group is a CTI-group; however, the converse is not true. In particular, the center of any non-nilpotent ATI-group is trivial, but this does not hold

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for CTI-groups. In this paper, we classify the CTI-groups with non-trivial center. Also we prove that these groups are necessarily solvable with elementary abelian center. Next, we determine the structure of solvable CTI-groups with trivial center, and show that the centralizers of their minimal normal subgroups are equal to the Fitting subgroup of the group. Also we prove that a CTI-group is solvable if and only if it has a solvable minimal normal subgroup. Finally we classify non-solvable CTI-groups.

Our notation is standard and can be found in [2] and [11]. Throughout this paper, $F(G)$ is the Fitting subgroup of G, $Z(G)$ is the center of G; also Q_8 and S_4 are the quaternion group of order 8, and the symmetric group of degree 4, respectively.

The following easy lemmas will be useful.

Lemma 1.1. *Let G be a* CTI-*group and H be a subgroup of G. Then:*

(i) *H is a* CTI-*group.*

(ii) *If H is cyclic and* $\mathrm{Core}_G(H) \neq 1$, *then $H \trianglelefteq G$.*

Lemma 1.2. *Let G be a* CTI-*group and assume that $x, y \in G$ have coprime orders. If $[x, y] = 1$ and $\langle x \rangle \trianglelefteq G$, then $\langle y \rangle \trianglelefteq G$.*

Proof. As $\langle x \rangle \leq \langle xy \rangle$, we have

$$\mathrm{Core}_G(\langle xy \rangle) \neq 1$$

and so $\langle xy \rangle \trianglelefteq G$. Now since $\langle y \rangle$ is a characteristic subgroup of $\langle xy \rangle$, we have $\langle y \rangle \trianglelefteq G$. □

As an immediate corollary, we get:

Corollary 1.3. *Let G be a* CTI-*group with non-trivial center.*

(i) *Assume that the order of $1 \neq g \in G$ is coprime to the order of an element of $Z(G)$. Then $\langle g \rangle \trianglelefteq G$.*

(ii) *If two distinct primes p and q divide the order of $Z(G)$, then G is a Hamiltonian group.*

Proof. (i) This is trivial.

(ii) Let $x \in G$ be of prime order r. Then, we have $(r, p) = 1$ or $(r, q) = 1$. Therefore by (i), $\langle x \rangle \trianglelefteq G$, consequently any cyclic subgroup of G and so any subgroup of G is normal in G (by Lemma 1.1 (ii)). □

The preceding corollary implies that a finite non-Hamiltonian nilpotent CTI-group is necessarily a non-abelian p-group.

2 CTI-groups with non-trivial center

In this section, we suppose that G is a non-nilpotent CTI-group with non-trivial center.

Theorem 2.1. *Let G be a non-nilpotent* CTI-*group with non-trivial center. Then $Z(G)$ is an elementary abelian p-subgroup, where p is the smallest prime divisor of $|G|$. In particular, any p'-subgroup of G is normal.*

Proof. Since G is not a Hamiltonian group, it follows that $Z(G)$ is a p-subgroup (by Corollary 1.3 (ii)). Also Corollary 1.3 (i) implies that any p'-subgroup of G is normal. Now it suffices to prove that every element of $Z(G)$ is of order p. Let $x \in Z(G)$ satisfy $x^{p^i} = 1$, where $i > 1$. Also assume that $\langle y \rangle \ntrianglelefteq G$ is of order p. As $\langle x^p \rangle \leq \langle yx \rangle$, we have $\langle yx \rangle \trianglelefteq G$. Therefore $\langle yx \rangle$ acts trivially on any p'-element t of G, and this implies that $[t, y] = [t, xy] = 1$. Now since $\langle t \rangle \leq \langle yt \rangle$, it follows that $\langle yt \rangle \trianglelefteq G$. Thus we conclude that $\langle y \rangle \trianglelefteq G$ which contradicts our assumption.

Now let q be the smallest prime divisor of $|G|$ and $q \neq p$. Let $y \in G$ be of order q. Then by Lemma 1.2, $\langle y \rangle \trianglelefteq G$. Consequently, $y \in Z(G)$. Hence we get a contradiction and the proof is complete. □

Remark 2.2. The preceding theorem states that a Hall p'-subgroup of any non-nilpotent CTI-group G with non-trivial center is Hamiltonian and normal, so we can write $G = HP$, where $P \in \mathcal{S}y\ell_p(G)$ and H is an abelian p'-subgroup, because $|H|$ is odd, since p is the smallest prime divisor or $|G|$. Also we immediately see that any non-normal cyclic subgroup is necessarily a p-subgroup.

We continue to assume that p is the smallest prime divisor of $|G|$.

Proposition 2.3. *Let G be a non-nilpotent* CTI-*group with non-trivial center. Then for every non-normal cyclic subgroup K of G, $\mathcal{C}_G(K)$ is a p-subgroup. In particular, $\mathcal{C}_H(P) = 1$ and accordingly $H \leq G'$.*

Proof. Let $K = \langle x \rangle$ and $y \in \mathcal{C}_G(x)$ be a p'-element. By Theorem 2.1, we have $\langle y \rangle \trianglelefteq G$. Lemma 1.2 implies that $\langle x \rangle \trianglelefteq G$ which contradicts our assumption. Therefore $\mathcal{C}_G(x)$ is a p-group and so we will have $\mathcal{C}_H(P) \leq \mathcal{C}_H(x) = 1$. Now the fundamental theorem of coprime actions implies that $H = [H, P]$ and hence $H \leq G'$. □

Theorem 2.4. *Let G be a non-nilpotent* CTI-*group with non-trivial center and p be the smallest prime divisor of $|G|$. If G has no subgroups isomorphic to a dihedral group of 2-power order, then any cyclic p-subgroup of order greater than p is non-normal.*

Proof. Let $\langle x\rangle \ntrianglelefteq G$ be of order p and let $y \in G$ satisfy $1 \neq y^p \in Z(G)$. If $p = 2$ and $(xy)^2 = 1$, then $y^x = y^{-1}$ and $\langle x, y\rangle$ is a dihedral group of 2-power order, which is contradiction. Thus

$$(xy)^p = y^p[y,x]^{\frac{p(p-1)}{2}},$$

since $[y,x] \in Z(G)$. Therefore $(xy)^p$ is a central element of G and so $\langle xy\rangle \trianglelefteq G$. Consequently, for any p'-element t, we have $[t,x] = [t,yx] = 1$ or $t \in \mathcal{C}_G(x)$ and this is in contradiction to Proposition 2.3. □

It follows from Theorem 2.4 that if a finite non-nilpotent CTI-group has no subgroups isomorphic to a dihedral group of 2-power order, then no power of any non-trivial element of its p-subgroups can be central.

We can now prove our main structural theorem:

Theorem 2.5. *Let G be a non-nilpotent* CTI-*group with non-trivial center and let p divide $|Z(G)|$. Then G possesses an abelian p-subgroup K such that*

$$P \cong K \rtimes \mathbb{Z}_{p^i}$$

and every subgroup of K is normal in G. Also,

(i) *if p is odd or P is an abelian subgroup, then*

$$K = Z(G) \quad \textit{and} \quad P = Z(G) \times \mathbb{Z}_{p^i},$$

also in this case $G' \cap Z(G) = 1$,

(ii) *if $p = 2$ and P is a non-abelian subgroup, then $i = 1$ and P has a subgroup isomorphic to a dihedral group of* 2*-power order, moreover $G' \cap Z(G) \neq 1$,*

(iii) $G' \cap Z(G) \neq 1$ *if and only if G possesses a subgroup isomorphic to a dihedral group of* 2*-power order.*

Proof. Let $h \in H$ with $|h| = q \neq p$. Then $\langle h\rangle \trianglelefteq G$ and P acts on $\langle h\rangle$ by conjugation, so there exists a homomorphism $\varphi : P \longrightarrow \mathrm{Aut}(\langle h\rangle)$.

Set $K := \ker\varphi$ and let $P/K = \langle xK\rangle$. Then $P = \langle x, K\rangle$. Clearly $\langle x\rangle \ntrianglelefteq G$, otherwise the action of x on h would be trivial. If for some i, $x^i \in K$ then we get $\langle x\rangle \trianglelefteq G$ and this is a contradiction. Thus $\langle x\rangle \cap K = 1$ and $P = K \rtimes \langle x\rangle$. As every element of K commutes with h, by applying Lemma 1.2, we conclude that every subgroup of K is normal in G and therefore K is a Hamiltonian group. Also it is clear that $Z(G) = \Omega_1(K)$.

(i) Let p be odd or P be an abelian group. Then G has no subgroup isomorphic to a dihedral groups of 2-power order. Thus Theorem 2.4 implies that any element of K is of order p and so $K = Z(G)$. Hence $P = Z(G) \times \mathbb{Z}_{p^i}$ and $G' = H$. Thus $G' \cap Z(G) = 1$.

(ii) First, we note that for any $y \in K$ and $1 \neq t \in \langle x \rangle$ we have $\langle yt \rangle \ntrianglelefteq G$; otherwise $[h, t] = [h, yt] = 1$ and so $t \in K \cap \langle x \rangle$, which is clearly a contradiction.

Let $y \in \mathcal{C}_K(x)$. If $|y| \neq 2$, then $(yt)^2 = y^2$, whence $t \in \langle x \rangle$ is a element of order 2. Therefore $\langle yt \rangle \trianglelefteq G$, a contradiction. Consequently, $Z(G) = \mathcal{C}_K(x)$.

Since P is non-abelian, we have $Z(G) \neq K$. Therefore, on assuming that $y \in K$ is of order 4 we see that $[y, x^2] = 1$ (since the action of $\langle x \rangle$ on $\langle y \rangle$ is at most of order 2). Now, if $|x| = l \neq 2$ then $y^2 \in \langle yx^{\frac{l}{2}} \rangle$ and so $\langle yx^{\frac{l}{2}} \rangle \trianglelefteq G$. This is a contradiction; consequently, $x^2 = 1$.

Now let $y \in K$ be an arbitrary element. Since $y^x \in \langle y \rangle$, we have $(yx)^2 \in K$. So, if $|yx| > 2$, then we get $\langle yx \rangle \trianglelefteq G$, a contradiction. Thus we have $|yx| = 2$ and $y^x = y^{-1}$, in other words, x inverts any element of K. Hence $\langle y, x \rangle$ is a dihedral group of 2-power order. So, $Z(\langle y, x \rangle) \leq G' \cap Z(G)$.

If K were a non-abelian group, then $Q_8 \leq K$, because K is a Hamiltonian group. Therefore K would contain two elements y and z of order 4 such that $|yz| = 4$ and $y^2 = z^2$. But in this case we would have

$$(yz)^{-1} = (yz)^x = y^x z^x = y^{-1} z^{-1} = (zy)^{-1}.$$

Thus $[z, y] = 1$ and so

$$(zy)^2 = z^2 y^2 = z^4 = 1,$$

a contradiction. Hence, K must be an abelian group.

(iii) First, let $G' \cap Z(G) \neq 1$. Then P is non-abelian. Therefore $K \neq Z(G)$, and so by (ii), G has a subgroup isomorphic to D_{2^l} for some l.

Conversely, assume that P has a subgroup isomorphic to D_{2^l}. In this case, by (ii), K has an element y of order 2^{l-1}, so $y^{2^{l-2}} \in Z(G)$ and also $y^{2^{l-2}} \in D'_{2^l}$. Hence, $G' \cap Z(G) \neq 1$. $\square$

Corollary 2.6. *Let G be a non-nilpotent* CTI-*group such that $Z(G) \neq 1$. Also suppose that p divides $|Z(G)|$ and let H be a Hall p'-subgroup of G. Then H is abelian and normal, and moreover $G = HP$ is solvable. Also,*

(i) *if $Z(G) \cap G' = 1$, then $G \cong K \times (H \rtimes \mathbb{Z}_{p^i})$, where p is the smallest divisor of $|G|$, $K = Z(G)$, $P = Z(G) \times \mathbb{Z}_{p^i}$ and $H = G'$,*

(ii) *if $Z(G) \cap G' \neq 1$, then $p = 2$ and $P = K \rtimes \mathbb{Z}_2$, where K is an abelian normal subgroup of G; also $Z(G) = \Omega_1(K)$, $G' = H\mho^1(K)$ and $\mathbb{Z}_2$ inverts any element of HK,*

(iii) *the Fitting subgroup $F(G) = HK$ is abelian.*

Lemma 2.7. *Let G be a non-nilpotent* CTI-*group with non-trivial center and let $\langle x \rangle \ntrianglelefteq G$. Then for any $y \in Z(G)$, $\langle x, y \rangle \ntrianglelefteq G$. So the center of any non-nilpotent* ATI-*group is trivial.*

Proof. Assume that $\langle x, y\rangle \trianglelefteq G$. Since any p'-subgroup is normal, it follows that x is a p-element. Therefore $\langle x, y\rangle \trianglelefteq G$ is a p-subgroup of G, and so x acts trivially on any p'-element of G. Now, by Lemma 1.2, $\langle x\rangle \trianglelefteq G$.

Since in every ATI-group, for any $y \in Z(G)$ and $g \in G$ we have $\langle y, g\rangle \trianglelefteq G$, and any ATI-group is a CTI-group, we get $\langle g\rangle \trianglelefteq G$ for every $g \in G$. Hence, G is Hamiltonian; a contradiction. □

3 Solvable CTI-groups with trivial center

In this section, we show that a CTI-group G is solvable if and only if it has a solvable minimal normal subgroup. Also assuming that G is a solvable group with trivial center we show that if V is a minimal normal subgroup of G, then $G \cong \mathcal{C}_G(V) \rtimes H$, where the Sylow subgroups of H are cyclic or isomorphic to Q_8 and $F(G) = \mathcal{C}_G(V)$. Also either $G \cong S_4$ or G is a Frobenius group with kernel $F(G)$ and complement H.

We remark that if a CTI-group G has a solvable minimal normal subgroup, then, by Corollary 2.6, every minimal normal subgroup of G is also solvable.

Suppose that V is a solvable minimal normal subgroup of G. As V is an elementary abelian p-subgroup, we have $V \leq F(G)$ and so $V \leq Z(F(G))$. Hence, $F(G) \leq \mathcal{C}_G(V)$.

Let $x \in \mathcal{C}_G(V)$. Then we have $V \leq \mathcal{C}_G(x)$. Now if $\mathcal{C}_G(x)$ is Hamiltonian, then $V \leq Z(\mathcal{C}_G(x))$ and so $\mathcal{C}_G(x) \leq \mathcal{C}_G(V)$. If $\mathcal{C}_G(x)$ is non-nilpotent and x is a p-element, then again $V \leq Z(\mathcal{C}_G(x))$ (by Corollary 2.6), and so $\mathcal{C}_G(x) \leq \mathcal{C}_G(V)$. In particular, as $\mathcal{C}_G(V) \leq \mathcal{C}_G(x)$ for any $x \in V$, we see that if $\mathcal{C}_G(x)$ is Hamiltonian or non-nilpotent, then $\mathcal{C}_G(x) = \mathcal{C}_G(V)$.

For the sake of simplicity in the following theorems we set $C_V = \mathcal{C}_G(V)$, $F = F(G)$ and $C_x = \mathcal{C}_G(x)$, for any $x \in G$.

Theorem 3.1. *Let G be a finite* CTI-*group with trivial center and V be a minimal normal subgroup of G. If V is solvable, then $F = C_V$.*

Proof. By the above discussion, it suffices to show that C_V is nilpotent. Suppose by way of contradiction that C_V is not nilpotent. Since $Z(C_V) \neq 1$, we conclude that $C_V \cong F \rtimes \mathbb{Z}_{p^i}$ where F is abelian. We claim that $C_x \leq C_V$ for any $x \in C_V$. Therefore G will be a Frobenius group with kernel C_V, and this is a contradiction, because C_V is not nilpotent.

Consider first the case $x \in Z(C_V)$. Then $C_V \leq C_x$. Therefore, C_x is also non-nilpotent and so $V \leq Z(C_x)$. Thus, $C_V = C_x$. Now assume that $x \notin Z(C_V)$. In this case, if $\langle x\rangle \trianglelefteq C_V$, then $x \in F(C_V) = F$ and so $F \leq C_x$. Also either x is a p'-element or $p = 2$ and $|x| = 2^l \neq 2$, so in either case, C_x is nilpotent by Theorem 2.1 and since it is not a p-group, it is a Hamiltonian group and $V \leq Z(C_x)$. Hence $F = C_x \leq C_V$.

Let $\langle x \rangle \ntrianglelefteq C_V$. If $|x| > p$, then C_x is necessarily nilpotent. Therefore by choosing $y \in V \cap Z(C_x) \neq 1$, C_y will be non-nilpotent because $C_V \leq C_y$. Thus we get $C_x \leq C_y = C_V$. Now if $|x| = p$, then either C_x is nilpotent and so we have $V \cap Z(C_x) \neq 1$, or C_x is non-nilpotent and hence $V \leq Z(C_x)$. So in either case, $C_x \leq C_V$. Thus C_V is nilpotent and so $F = C_V$. □

Notice that the Fitting subgroup of a CTI-group is not necessarily abelian. For example, using the Small Group library of GAP, we see that the group SmallGroup(9477,4035), is a CTI-group with abelian center and non-abelian Fitting subgroup. The structure of this group is as follows:

$$G \cong ((\mathbb{Z}_3 \times \mathbb{Z}_3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_{13},$$

and its Fitting subgroup is $F(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_3$.

If the order of $F(G)$ is divisible by more than one prime, then $F(G)$ is abelian.

Proposition 3.2. *Let G be a finite* CTI-*group with trivial center and also let its minimal normal subgroup be solvable. If $|F|$ has more than one prime divisor, then $G = FH$ is a Frobenius group with abelian kernel F and complement H.*

Proof. By Corollary 1.3 (ii), F is a Hamiltonian group. Therefore $F' \leq Z(G) = 1$ and so F is an abelian group.

Assume that q is a prime divisor of $|F|$ and $Q \in \mathcal{S}y\ell_q(G)$. As $F \cap Q \trianglelefteq Q$, we have $F \cap Z(Q) \neq 1$. Consequently, on assuming $x \in F \cap Z(Q)$, C_x contains both F and Q. Next, we show that F is a Hall subgroup of G. First we assume that C_x is nilpotent. Since $Q \trianglelefteq C_x$, Q commutes with a minimal normal subgroup V of order coprime to q. Thus, $Q \leq C_V = F$.

Now, let C_x be non-nilpotent. By Lemma 2.1, C_x contains a minimal normal subgroup V of q-power order. Also, since V is elementary abelian, it follows that $V \leq Z(C_x)$, therefore $Q \leq C_x \leq C_V = F$. Thus, F is a Hall subgroup of G. Consequently, $G = FH$.

Finally, to complete the proof it will suffice to show that for every $x \in F$, $C_x \leq F$. Let q be a prime divisor of $|C_x|$ such that $q \nmid |F|$. Also let $y \in C_x$ be of order q. If C_x is nilpotent, then $y \in C_G(F) = F$ and this is a contradiction. Now, let C_x be non-nilpotent. Then since x and y have coprime orders, Corollary 2.6 (iii) implies that $y \in F(C_x)$ and $F(C_x)$ is abelian. So again $y \in C_G(F) = F$, because $F \leqslant F(C_x)$, which gives the final contradiction. Hence, $C_x = F$ completing the proof. □

In the following theorems, we suppose that F is a p-group.

Lemma 3.3. *Let G be a* CTI*-group with trivial center and $K \leq G$. Also assume that a minimal normal subgroup of G is solvable and F is a p-group. Then:*

(i) *for any $x \in F$, C_x is a p-group,*

(ii) *if $P \in \mathcal{S}y\ell_p(G)$ is maximal in K and $P \ntrianglelefteq G$, then K is a non-nilpotent group with trivial center. Also, $F(K)$ is a p-subgroup of K and $P \ntrianglelefteq K$.*

Proof. (i) Let V be a minimal normal subgroup of G and $x \in F$. Suppose that C_x is not a p-group. Since any p'-subgroup of C_x is normal, whether C_x is or is not nilpotent, we see that $F = C_V$ contains a p'-element (because $V \leqslant C_x$) and this is a contradiction. Hence for any $x \in F(G)$, we observe that C_x is a p-group.

(ii) Suppose $K \leq G$ contains P as a maximal subgroup. Then $V \leq F \leq F(K)$. Now since for every $x \in F$, the subgroup C_x is a p-group, so is $F(K)$. Therefore, $F(K) = \mathrm{Core}_G(P) = F$. Thus K is non-nilpotent and also $Z(K) = 1$ (otherwise, since $Z(K) \leq F$, for any $x \in Z(K)$, $K \leq C_x$ would be a p-group). □

Theorem 3.4. *Let G be a finite solvable* CTI*-group with trivial center. Assume further that F is a p-group. Then either G is isomorphic to S_4, or F is a Sylow p-subgroup of G and G is a Frobenius group with kernel F.*

Proof. Let P be a Sylow p-subgroup of G. If P is normal in G, then $F = P$ is the Frobenius kernel and the desired conclusion follows. So let $P \ntrianglelefteq G$. We shall show $G \cong S_4$.

Assume now that P is a maximal subgroup of $K \leq G$. By the preceding lemma, we have $Z(K) = 1$ and $P \ntrianglelefteq K$. Now, if the conclusion is established for K namely, $K \cong S_4$, then $F \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, we get $S_3 \cong K/F \leq G/F \hookrightarrow S_3$, therefore $K = G$. Hence without loss of generality we may assume that P is maximal in G.

Let Q be a Sylow q-subgroup of G, whence $q \neq p$. Then QF is a Frobenius group with kernel F. Therefore Q is either cyclic or generalized quaternion. As P is a maximal subgroup of G, we have $G = PQ$, furthermore, QF/F is a unique minimal normal subgroup of G/F, because $F = \mathrm{Core}_G(P)$. Hence we will have $Q \cong \mathbb{Z}_q$ and so $q \neq 2$ (otherwise, $P \trianglelefteq G$). Also, $P/F \hookrightarrow \mathrm{Aut}(Q)$. Thus P/F is cyclic and $p|q-1$.

Now, set $N = \mathcal{N}_G(Q)$. Then by the Frattini argument, we have $G = NF$, because $QF \trianglelefteq G$. If $F \cap N \neq 1$, then since $[F \cap N, Q] = 1$, we will have $Q \leq C_x$, for any $x \in N \cap F$ and this is a contradiction, since C_x is a p-group. Thus, we obtain $F \cap N = 1$ and so $Q \lneqq N$. Let P_1 be a Sylow p-subgroup of N. Then P_1 is cyclic and $N = QP_1$ is a CTI-group. As $FZ(N) \trianglelefteq G$, we have

$$Z(N) \leq F \cap N = 1$$

so $\mathrm{Core}_N(P_1) = 1$, therefore $|P_1| \mid q-1$.

Assume that V is a minimal normal subgroup of G and also a and x are generators of P_1 and Q, respectively.

Step 1. $\mathcal{C}_F(a) \cap (\mathcal{C}_F(a))^x = 1$ and so $Z(P) \cap Z(P^x) = 1$.

Assume that $f \in \mathcal{C}_F(a) \cap (\mathcal{C}_F(a))^x$. Then there exists an element $f_1 \in \mathcal{C}_F(a)$ such that $f = f_1^x$. Therefore $f_1^x = (f_1^x)^a = f_1^{x^a}$ and so $f_1 \in \mathcal{C}_F([x,a]) = 1$, because $[x,a] \in Q$.

Step 2. $p = 2$ and $|(VP_1)'| = |P_1| = 2$.

Let $|P_1| = p^m$ and $z \in Z(P) \cap V$ be of order p. We set $z_i = z^{x^i}$, for any $i \geq 0$. Then $C = \{z_i \mid 0 \leq i < q\}$ is the set of conjugates of z by Q. The set C is also invariant under conjugation by P_1 and if for some $l \neq 0$ and $i > 1$, $z_i^{a^l} = z_i$, then $z^{x^i} = z^{a^{-l}x^i a^l}$. Thus

$$a^{-l}x^i a^l x^{-i} \in \mathcal{C}_Q(z) = 1,$$

so $a^{-l}x^i a^l = x^i$ then $a^l \in \mathrm{Core}_N(P_1) = 1$, which is a contradiction. Consequently, only the element $z = z_0$ of C is invariant under the action of P_1. Therefore, we have

$$C = \{z\} \cup \bigcup_{l=1}^{k} \mathrm{Orbit}_{P_1}(z_{i_l}).$$

Now, let $u = \prod_{i=0}^{q-1} z_i$. Since $u^x = u$, we have $u \in \mathcal{C}_F(x) = 1$. Thus

$$1 = \prod_{i=0}^{q-1} z_i = z \prod_{l=1}^{k} \prod_{t \in \mathrm{Orbit}_{P_1}(z_{i_l})} t. \tag{$*$}$$

If $\exp(VP_1) = p^m$, then

$$1 = (a^{-1}z_i)^{p^m} = \prod_{l=1}^{p^m} z_i^{a^l} = \prod_{t \in \mathrm{Orbit}_{P_1}(z_i)} t.$$

By $(*)$, $z = 1$ and this is a contradiction. Thus there exists a $z_i \in C$ such that $a^{-1}z_i$ is of order p^{m+1}. Since $a^{-1}z_i \notin V$, it follows that $v = (a^{-1}z_i)^{p^m}$ belongs to the center of VP_1, therefore $\langle a^{-1}z_i \rangle \trianglelefteq VP_1$ (VP_1 is a CTI-group). Also we will have

$$VP_1/\langle v \rangle \cong V/\langle v \rangle \times \langle a^{-1}z_i \rangle / \langle v \rangle.$$

Thus $[VP_1, VP_1] = \langle v \rangle \leq Z(VP_1)$ and so

$$(az_i)^p = a^p z_i^p [z_i, a]^{p(p-1)/2}.$$

If p is odd or $m > 1$, then we have $(az_i)^{p^m} = a^{p^m} = 1$ and this a contradiction. Hence, $p = 2$, $m = 1$ and $|P_1| = 2$.

Step 3. $V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $q = 3$ and $VN \cong S_4$.

We set $Z = Z(VP_1)$. Then $Z \cap Z^x = 1$ by step 1. Since $C \subseteq Z(F(G))$, we have $\langle C \rangle \trianglelefteq G$ therefore $V = \langle C \rangle$. Since for any $i > 1$, $[z_1, a] = [z_i, a]$, it follows that $z_1 z_i^{-1} \in Z$; consequently, $V/Z \cong \langle z_1 \rangle$ and so $Z^x \cong \mathbb{Z}_2$ and $V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence $q = 3$ and $VN \cong S_4$.

Step 4. $F(G)$ is the unique minimal normal subgroup of G and thus $G \cong S_4$.

Let z_1 and z_2 be two distinct central elements of order 2. Then for $v_1 = z_1^x$ and $v_2 = z_2^x$, the subgroups $V_1 = \langle z_1, v_1 \rangle$ and $V_2 = \langle z_2, v_2 \rangle$ will be two distinct minimal normal subgroup of G. Thus $v_1^a = z_1 v_1$ and $v_2^a = z_2 v_2$, and also

$$(av_1)^{v_2} = v_2 a v_1 v_2 = a v_1 z_2.$$

Since P is a CTI-group and $(av_1)^2 = (av_1 z_2)^2 = z_1$, we will have

$$av_1 z_2 = (av_1)^3 = av_1 z_1$$

and so $z_1 = z_2$, a contradiction. Thus $Z(P)$ is cyclic and therefore G possesses a unique minimal normal subgroup $\langle z, v \rangle$, where $z \in Z(P)$ and $v \in V$.

As $(va)^2 = z$, we have $\langle va \rangle \trianglelefteq P$ and so $[F, \langle va \rangle] \leq F \cap \langle va \rangle = \langle z \rangle$. Since for every $f \in F$, $[f, v] = 1$, we will have $[F, a] \leq \langle z \rangle$ and so $F^2 \leq \mathcal{C}_F(a)$; consequently, $\mathcal{C}_F(a) \trianglelefteq F$ and $F/\mathcal{C}_F(a)$ is elementary abelian.

Finally assume that $f_1, f_2 \notin \mathcal{C}_F(a)$. Then we have $f_2^{-1} f_1 \in \mathcal{C}_F(a)$, because $[f_1, a] = [f_2, a]$. Therefore, $F/\mathcal{C}_F(a)$ is cyclic and so it is isomorphic to $\mathbb{Z}_2$. By step 1, we have $|\mathcal{C}_F(a)| = |\mathcal{C}_F(a)^x| = 2$, consequently, $F = V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and the desired conclusion follows. □

Theorem 3.5. *Let $G = KH$ be a finite Frobenius* CTI-*group with kernel K and complement H. Then,*

(i) *if $|H|$ is odd, then H is cyclic,*

(ii) *if $|H|$ is even, then K is abelian and either H is cyclic or $H \cong Q_8 \times \mathbb{Z}_n$, where n is odd.*

In either case G is solvable.

Proof. (i) Since H is a solvable group and cannot be Frobenius group by [10, Theorem 12.6.11], it follows that $Z(H) \neq 1$ by Theorem 3.4 and 3.2. Now by Corollary 2.6, H is a nilpotent. Therefore H is cyclic by [2, Theorem 10.3.1 (iv)].

(ii) By [2, Theorem 10.3.1 (iii), (iv)], K is abelian and $Z(H) \neq 1$ again by Corollary 2.6, H is nilpotent. We can easily see that the only generalized quaternion CTI-group is Q_8. Therefore either H is a cyclic group or $H \cong Q_8 \times \mathbb{Z}_n$, where n is odd. □

Theorem 3.6. *A* CTI-*group G is solvable if and only if it has a solvable minimal normal subgroup.*

Proof. If $Z(G) \neq 1$ or $F(G)$ is not a p-group, then by Proposition 3.2 and Corollary 2.6, G is solvable. So we assume that $Z(G) = 1$ and $F(G)$ is a p-group.

Let G be a minimal counterexample for the theorem. Let $P \in \mathcal{S}\mathrm{y}\ell_p(G)$. By Theorems 3.5 and 3.4, $P \ntrianglelefteq G$. Suppose that a proper subgroup K of G contains P as a maximal subgroup. Therefore we have $P \ntrianglelefteq K$, $F(K) = F(G)$ and $Z(K) = 1$ (by Lemma 3.3), also by the choice of G, K is solvable and so $K \cong S_4$. Hence $P \cong D_8$ and $F(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore $G/F(G)$ is solvable which is a contradiction. And so P is a maximal subgroup of G. By a well-known theorem of Thompson [2, Theorem 10.3.2], $p = 2$ and by [9, Theorem II], G/F has a unique minimal normal subgroup K/F such that G/K is a 2-group. Hence K is not solvable. Again by the minimality of G, we have $K = G$. Now by [5, Theorem 2.13] every involution of G/F inverts an element of odd order in G/F, so G/F contains a non-nilpotent dihedral subgroup. Consider the inverse image R of this dihedral subgroup in G. Obviously $Z(R) = 1$ and R is solvable with non-normal Sylow 2-subgroup. By using Theorem 3.4, $R \cong S_4$ and F is a four group and this is also a contradiction. □

4 Non-solvable CTI-groups

In this section we classify non-solvable CTI-groups. Let V be a minimal normal subgroup of a non-solvable CTI-group G. By Theorem 3.6, V cannot be solvable, since the centralizer of any element (in particular any subgroup) of G is solvable, and so $\mathcal{C}_G(V) = 1$. Therefore, V must be simple. Also we have

$$V \leq G \hookrightarrow \mathrm{Aut}(V) \quad \text{and} \quad G/V \hookrightarrow \mathrm{Out}(V).$$

Lemma 4.1. *Let G be a non-solvable* CTI-*group with minimal normal subgroup V and $P \in \mathcal{S}\mathrm{y}\ell_2(V)$. If $N = \mathcal{N}_G(P)$ is non-nilpotent, then $Z(N) = 1$.*

Proof. If $Z(N) \neq 1$, then by Corollary 2.6 either $P \leqslant Z(N)$ or $\mathcal{C}_G(P)$ has index 2 in N. In the latter case, we have $\mathcal{N}_V(P) = \mathcal{C}_V(P)$. In either case, we get $P \leqslant Z(\mathcal{N}_V(P))$ and so P has a normal p-complement in V, a contradiction. □

Theorem 4.2. *Let G be a finite non-solvable* CTI-*group. Then $G \cong \mathrm{PSL}(2,q)$ or $G \cong \mathrm{PGL}(2,q)$, where $q > 3$ is a prime power.*

Proof. Let G be a finite non-abelian simple CTI-group. Since every p-local subgroup of G is solvable, then G is an N-group. Now by a theorem of Thompson ([2, Theorem, p. 474]), only the groups $\mathrm{PSL}(2,q)$ and $\mathrm{Sz}(q)$ which do not contain $\mathrm{SL}(2,3)$ can be CTI (because $\mathrm{SL}(2,3)$ is not a CTI-group). Let $G \cong \mathrm{Sz}(q)$ and $P \in \mathcal{S}\mathrm{y}\ell_2(G)$. Then by [1, Lemma 1 and Proposition 3] we have $\Omega_1(P) = Z(P)$

and $|P| = |Z(P)|^2$. Since P is a non-abelian CTI-group, P must be a non-abelian Hamiltonian group of order 16. This is a contradiction.

Now we consider the non-simple case: then G is isomorphic to a subgroup of $H = \mathrm{Aut}(\mathrm{PSL}(2,q)) = \mathrm{PGL}(2,q) \rtimes \langle x \rangle$, where $q = p^f$ and x has order f. Let $g \in G \setminus \mathrm{PGL}(2,q)$ be a power of x. Then $f \neq 1$ also $\mathrm{PSL}(2,p) \leqslant \mathcal{C}_G(g)$, because $\mathcal{C}_H(x) = \mathrm{PGL}(2,p) \times \langle x \rangle$. Since $\mathcal{C}_G(g)$ is non-Hamiltonian and solvable, it follows that $|g| = 2$ (by Corollary 2.6), and $p = 2$, because a Sylow 3-subgroup of $\mathrm{PSL}(2,3)$ is non-normal. Now let $S \in \mathcal{S}y\ell_2(G)$ and $P \in \mathcal{S}y\ell_2(\mathrm{PGL}(2,q))$ such that $P \leqslant S$. Then $S = P\langle g \rangle$. Suppose $N = \mathcal{N}_G(P)$; by Lemma 4.1, $Z(N) = 1$. If $S \trianglelefteq N$, then $N = S\langle y \rangle$, where $|y| = q-1$ (by [2, Lemma 15.1.1]). Hence $[g,y] = 1$ and N cannot be a Frobenius group; now by Theorem 3.4, $N \cong S_4$ and $f = 2$. Therefore, $G \cong \mathrm{Aut}(\mathrm{PSL}(2,4))$ which is isomorphic to $\mathrm{PGL}(2,5)$.

In the other case, since G is a pre-image of a subgroup of

$$\mathrm{Out}(\mathrm{PSL}(2,q)) = \langle \bar{y} \rangle \times \langle \bar{x} \rangle, \quad \text{where } |y| = (2, q-1),$$

then either G is isomorphic to $\mathrm{PGL}(2,q)$, where $q > 3$ is a prime power or p is odd, f is even and $G \cong \langle \mathrm{PSL}(2,q), yx^{f/2} \rangle$. In the latter case G is isomorphic to a non-solvable maximal subgroup of $\mathrm{PGL}(2,q) \rtimes \langle x^{f/2} \rangle$. Now by [3, Lemma 6.6.3], G is isomorphic to $\mathrm{PGL}^*(2,q)$ which has semidihedral Sylow 2-subgroup. This case cannot occur because a semidihedral group is not CTI. □

The inverses of Corollary 2.6 and Theorem 3.4 are simple: we just prove the inverse of the non-solvable case. Before proving the inverse theorem, we consider the simple fact that if a non-normal subgroup $\langle x \rangle$ of G is normal in a non-normal maximal subgroup M, then $\langle x \rangle \cap \langle x \rangle^g \trianglelefteq G$, where $g \in G \setminus M$.

Theorem 4.3. *Let G be isomorphic to K, where* $\mathrm{PSL}(2,q) \leq K \leq \mathrm{PGL}(2,q)$, *$q > 3$ is a power of prime p. Then G is a* CTI-*group.*

Proof. We can simply check by GAP that $\mathrm{PSL}(2,p)$ is CTI for $p = 5, 7, 9, 11$. Let x be an element of G. If $p \mid |x|$, then x must be a p-element, because by [2, Lemma 15.1.1] Sylow p-subgroups of G are elementary abelian and TI; therefore $|x| = p$. If $|x| \mid (q^2-1)$ and x is not a 2-element, then $|x| \mid 2^n m$, where m is odd; hence $x = yz$, where $|z| > 1$ is odd. In this case z belongs to the maximal subgroup $D_{2(q-1)}$ or $D_{2(q+1)}$ by [7, Theorem 2.1 and Theorem 2.2]; since $\langle z \rangle$ is normal in these groups, it follows that $\mathcal{N}_G(x) = \mathcal{N}_G(z)$ is a non-normal maximal subgroup of G. Therefore, $\langle x \rangle$ is normal in a non-normal maximal subgroup of G, and so is TI. Now, let x be a 2-element and $|x| > 2$; then p is an odd prime and again $\langle x \rangle$ belongs to the dihedral group. Since $\langle x \rangle$ is normal in this group, it follows that $\mathcal{N}_G(x)$ is maximal in G. Hence $\langle x \rangle$ is a TI-group. Therefore, G is a CTI-group. □

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